Fluctuations and entropy driven space–time intermittency in Navier–Stokes fluids.

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Abstract: *We analyze the physical meaning of fluctuations of the phase space contraction rate, that we also call entropy creation rate, and its observability in space–time intermittency phenomena. For concreteness we consider a Navier–Stokes fluid.*

§1. The chaotic hypothesis in turbulence.

Consider a Navier–Stokes (NS) fluid in a container V which we take, for simplicity, cubic with periodic boundary conditions and subject to a constant volume force $f\,\underline{\varphi}\,(\underline{x})$ with $\max|\,\underline{\varphi}\,(\underline{x})| = 1$ and with only Fourier harmonics corresponding to large wavelength of the order of the linear size L of V.

The viscosity will be denoted ν, but it is convenient to rescale space, time, velocity and pressure to write the equations in dimensionless form as

$$\dot{\underline{u}} + R\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u} = \Delta\,\underline{u} - \underline{\partial}\,p + \underline{\varphi}\,, \qquad \underline{\partial}\cdot\underline{u} = \underline{0}\,, \qquad R = fL^3\nu^{-2} \tag{1.1}$$

in a container of side $L = 1$, where R is the *Reynolds number*. We can suppose that $\int_V \underline{u}\,d\underline{x} = \underline{0}$ (because of translation invariance).

We assume the chaotic hypothesis

Chaotic hypothesis: *Asymptotic motions of a turbulent flow develop on an attracting set $\mathcal{A}$ in phase space on which time evolution $\underline{u} \to S_t\,\underline{u}$ can be regarded as a transitive Anosov system for the purposes of computing time averages in stationary states.*

Here we investigate some assumptions under which the hypothesis acquires some non trivial predictive value with implications that can have experimental relevance. For earlier reviews on the chaotic hypothesis see [Ga98a], [Ga96d], [Ga99a]. A recent one is [Ru99a].

§2. The OK41 cut–off.

Anxiety often mars the beginning of any discussion on the NS equations: it is a fact that to date there is *no theory* that allows a constructive solution of the equations via a controlled approximation scheme. Nevertheless most people rapidly recover and adopt the viewpoint that “physically there is an effective ultraviolet cut–off” and the NS equations can be reduced to ordinary equations:

The OK41 cut–off hypothesis: *There exists $\kappa_0 > 0$ such that if the NS equation, (1.1), is truncated in momentum space at $K(R) = R^{\kappa_0}$ (or higher) then the physically relevant predictions are not affected.*

The OK41 theory, see [LL71], assigns to κ_0 the value $3/4$. Therefore the flows of physical interest should be described by (1.1) truncated at $|\,\underline{k}\,| \le K(R)$, *i.e.*

$$\dot{\underline{u}}_{\,\underline{k}} + i\,R \sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\,\underline{k}_j|\le K(R)}} \underset{\sim}{u}_{\,\underline{k}_1}\cdot\underset{\sim}{k}\,\Pi_{\underline{k}}\,\underline{u}_{\,\underline{k}_2} = -\,\underline{k}^{\,2}\,\underline{u}_{\,\underline{k}} + \underline{\varphi}_{\,\underline{k}} \tag{2.1}$$

where $\underline{u}(\underline{x}) \stackrel{def}{=} \sum_{\underline{k}\ne\underline{0}} e^{i\,\underline{k}\cdot\underline{x}}\,\underline{u}_{\,\underline{k}}$ and $\Pi_{\underline{k}}$ is the projection orthogonal to $\underline{k}$; $\underline{k} = 2\pi\,\underline{n}$ with $\underline{n}$ an integer components vector.

Equation (2.1) admits an *a priori* bound on the energy $\dot{E}/2 = -\sum_{\underline{k}} \underline{k}^2|\underline{u}_{\,\underline{k}}|^2 + \sum_{\underline{k}} \overline{\varphi}_{\underline{k}}\cdot\underline{u}_{\,\underline{k}}$ which implies that asymptotically in time the energy is bounded by $E \le 2||\underline{\varphi}||^2/(2\pi)^4$.

We shall call μ_R the "*statistics*" of the NS equation defined as the probability distribution on *phase space* (*i.e.* on the space of the velocity fields $\{\underline{u}_{\,\underline{k}}\}$, $|\underline{k}| < K(R)$) which describes, at Reynolds number R, the stationary state averages of observables $F(\underline{u})$, *i.e.* the probability distribution such that

$$\lim_{T\to\infty} T^{-1}\int_0^T F(S_t\,\underline{u})dt = \int F(\underline{v})\mu_R(d\,\underline{v}) \tag{2.2}$$

for almost all initial data $\underline{u}$. The distribution μ_R exists and is unique because of the chaotic hypothesis and it is also called the SRB distribution of the stationary state of (2.2).

3. NS and GNS equations: viscosity and vorticity ensembles. Equivalence.

For the purposes of a conceptual analysis stressing the analogy with the theory of ensembles in statistical mechanics we temporarily introduce a *control parameter* λ in front of the Laplacian in (1.1) and in front of the $-\underline{k}^2\,\underline{u}_{\,\underline{k}}$ term in (2.1): bearing in mind, however, that we are interested in $\lambda = 1$ *only.* The stationary distributions will then depend on λ, R and will be denoted $\mu_{\lambda,R}$ so that the previously introduced SRB distribution μ_R, (1.1), is $\mu_R \equiv \mu_{1,R}$ with the new notations.

The collection $\mathcal{E}_{NS}$ of all the probability distributions $\mu_{\lambda,R}$ is the collection of all the stationary states of the fluid, at varying Reynolds number R. It is an *ensemble* in the sense of statistical mechanics and it will be called the "*viscosity ensemble*" for the NS equations.

The second idea is that the same fluid can be studied, rather than by the NS equations (2.1), by considering the Euler equations subject to a dissipation mechanism that keeps the *vorticity* $\mathcal{S} \stackrel{def}{=} \int_V (\underset{\sim}{\partial}\,\underline{u})^2\,d\,\underline{x}$ bounded. This "thermostatting" effect can be achieved by imposing various types of forces $\underline{th}$ on the system so that $\mathcal{S} =$ *const* leading to

$$\dot{\underline{u}} + R\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u} = -\,\underline{\partial}\,p + \underline{\varphi} + \underline{th}\,, \qquad \underline{\partial}\cdot\underline{u} = \underline{0}, \qquad R = fL^3\nu^{-2} \tag{3.1}$$

As an example we can consider the force obtained by imposing the constraint that $\mathcal{S}$ remains identically constant via Gauss' principle of least effort, *c.f.r.* [Ga96b], appendix. It corresponds to, *c.f.r.* [Ga96b]

$$\dot{\underline{u}} + R\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u} = -\,\underline{\partial}\,p + \underline{\varphi} + \nu_G(\underline{u})\,\Delta\,\underline{u}\,, \qquad \underline{\partial}\cdot\underline{u} = \underline{0} \tag{3.2}$$

where $\nu_G(\underline{u})$ is an easily determined multiplier defined so that $\mathcal{S}$ is *exactly* conserved, namely

$$\nu_G(\underline{u}) = \frac{\int_V \big(\underline{\varphi}\cdot\Delta\,\underline{u} - R\Delta\,\underline{u}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})\big)\,d\,\underline{x}}{\int_V(\Delta\,\underline{u})^2\,d\,\underline{x}} \tag{3.3}$$

where V is the container region.

The collection $\tilde{\mathcal{E}}$ of the statistics $\tilde{\mu}_{\mathcal{S},R}$ for (3.1) will be called the "*vorticity ensemble*". We establish a correspondence between elements of the ensembles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ by saying that two elements $\mu_{\lambda,R} \in \mathcal{E}$ and $\mu_{\mathcal{S},R} \in \tilde{\mathcal{E}}$ are correspondent if

$$\tilde{\mu}_{\mathcal{S},R}(\nu_G) = \lambda \tag{3.4}$$

We shall call "*local*" an observable $F(\underline{u})$ that depends only on finitely many Fourier components of the velocity field $\underline{u}$ (this is *locality in momentum space*) and $\mathcal{L}$ is the family of the local observables. The analogy with statistical mechanics is quite manifest if the following conjecture holds, [Ga95b],

Equivalence conjecture: *Let $\mu_{\lambda,R}\in\mathcal{E}$ and $\mu_{\mathcal{S},R}\in\tilde{\mathcal{E}}$ be corresponding elements of the viscosity and vorticity ensembles, i.e. if $\mathcal{S}$ and λ are related by (3.4), then it is*

$$\lim_{R\to\infty}\frac{\tilde\mu_{\mathcal{S},R}(F)}{\mu_{\lambda,R}(F)}=1 \tag{3.5}$$

for all local observables $F\in\mathcal{L}$ with non zero average.

In other words the statistics of the *irreversible* NS equation (1.1) and of the *reversible* "Gaussian NS equation" (3.1), called GNS equation, form two *equivalent ensembles of stationary states* of the fluid. By "reversible" we mean that there is an involutory map I of phase space which *anticommutes with time evolution*, *i.e.*

$$I^2=1,\qquad I\,S_t=S_{-t}I\quad\text{for all}\quad t \tag{3.6}$$

and the GNS equations are reversible because $\nu_G(\underline{u})$ is odd under the transformation $I\,\underline{u}(\underline{x})=-\underline{u}(\underline{x})$. A similar conjecture has been proposed in certain models of non equilibrium statistical mechanics, [Ga96b], [Ru99b], [Ga99d].

For reference purposes we write explicitly the GNS equations with the OK41 cut–off

$$\dot{\underline{u}}_{\underline{k}}+i\,R\sum_{\substack{\underline{k}_1+\underline{k}_2=\underline{k}\\ |\underline{k}_j|\le K(R)}}\underline{u}_{\underline{k}_1}\cdot\underline{k}\,\Pi_{\underline{k}}\,\underline{u}_{\underline{k}_2}=-\nu_G(\underline{u})\,\underline{k}^2\,\underline{u}_{\underline{k}}+\underline{\varphi}_{\underline{k}} \tag{3.7}$$

with the same notations of (2.1).

The analogy with equilibrium theory of ensembles is: the parameter R plays the role of the *volume* while λ that of the *temperature* and $\mathcal{S}$ that of the *energy*. Therefore the viscosity ensemble is the analogue of the *canonical ensemble* and the vorticity ensemble is the analogue of the *microcanonical ensemble*. The $R\to\infty$ is analogous to the "*thermodynamic limit*", [Ga99a].
We see also why it is useful to introduce the parameter λ: if we stick to $\lambda=1$ then effectively we consider only a *single* stationary state $\mu_{1,\lambda}=\mu_R$ and *not* an ensemble: this state is "the same" (in the sense (3.5)) as the state $\tilde\mu_{\mathcal{S}_R,R}$ if $\mathcal{S}_R$ is so defined that $\tilde\mu_{\mathcal{S}_R,R}(\nu_G)=1$. The parameter λ will be set to its physical value 1 from now on.

The conjecture of equivalence was proposed in [Ga97a] and discussed in several other papers, see for instance [Ga97b]. It has been investigated by simulations in [RS99] with results that seem moderately satisfactory.

§4. Time reversal and fluctuation theorem.

We now consider the NS equation (3.7) and try to find some of its properties under the assumption that it is equivalent to the corresponding GNS equation, *i.e.* (3.7) with $-\underline{k}^2\,\underline{u}_{\underline{k}}$ replaced by $-\nu_G(\underline{u})\,\underline{k}^2\,\underline{u}_{\underline{k}}$. We assume the chaotic hypothesis and the OK41 cut–off and furthermore that

Transitivity and axiom C: *Either the full ellipsoid in phase space*

$$\{\,\underline{u}\,|\sum_{|\underline{k}|<R^{\kappa_0}}\underline{k}^2|\,\underline{u}_{\underline{k}}|^2=\mathcal{S}_R\} \tag{4.1}$$

is densely visited by the evolutions of all data starting on it apart from a zero volume set, or alternatively the evolution on this ellipsoid verifies a geometric property called "axiom C".

Axiom C says that if the system is not transitive because there is an attracting set $\mathcal{A}$ that is *smaller* than the full phase space (*i.e.* the ellipsoid (4.1) in this case) then, considering the simple case in which this happens because in phase space there are just a non dense attracting set and a repelling set (also not dense),

(1) the attracting and the repelling sets are smooth manifolds and all their points, but a set of zero surface area, generate dense trajectories on them, and
(2) the stable manifold of the points on the attracting set crosses transversally the repelling set and viceversa the unstable manifold of a point on the repelling set crosses transversally the attracting set.

for details, which will not be really necessary here, we refer to [BG98]. This implies that either the system is transitive or that its restriction to the attracting set is transitive.
The interest of the Axiom C notion is that it is a geometric property that has a remarkable consequence for systems admitting a time reversal symmetry I but with an attracting set $\mathcal{A}$ *which is not* the full phase space and, therefore, is mapped by I onto a repelling set $I\,\mathcal{A}$ different from $\mathcal{A}$. If the axiom holds one can define, [BG98], a map $P:\,\mathcal{A}\longleftrightarrow I\,\mathcal{A}$, of the attracting set $\mathcal{A}$ on the repelling set $I\mathcal{A}$ which commutes with time evolution and with I and

$$I\,P\,S_t = S_{-t}\,I\,P \tag{4.2}$$

i.e. the restriction of the transitive evolution S_t to the attracting set $\mathcal{A}$ *is still reversible*, although it is such for a *new time reversal operation*, namely $I\,P$, see [BG98], [Ga98b].

If a reversible evolution verifies axiom C and depends on a parameter and, as the parameter varies, it develops an attracting set $\mathcal{A}\neq I\,\mathcal{A}$ that is not the full phase space then the restriction of the evolution to the attracting set is time reversible with respect to a new time reversal symmetry.
In other words in axiom C systems time reversal symmetry I *cannot be really broken*: if there is a spontaneous breakdown (such has to be considered the "breaking", as a parameter varies, of phase space into an attracting set $\mathcal{A}$ smaller than phase space and a repelling set $I\,\mathcal{A}$ different from $I\,\mathcal{A}$, [Ga98b]) a new time reversal PI is "spawned".

The axiom C property is stable under perturbations: changing slightly parameters a system keeps this property if it has it to begin with. The transitivity (or axiom C) property is relevant because of the following theorem

Theorem *(fluctuation theorem): Let $-\sigma(\,\underline{u}\,)$ be the divergence of the GNS equations (3.7) and let $-\langle\sigma\rangle_+$ be its stationary average with respect to the SRB distribution.*
Then the (dimensionless) quantity

$$p=\tau^{-1}\int_{-\tau/2}^{\tau/2}\frac{\sigma(S_t\,\underline{u}\,)}{\langle\sigma\rangle_+}\,dt \tag{4.3}$$

*which we call "*average over a time span τ of the (dimensionless) phase space contraction at $\underline{u}$*" has a probability of being in the interval $[p,p+dp]$ of the form $\pi_\tau(p)\,dp = const\,e^{\zeta(p)\tau+O(1)}$ with*

$$\zeta(-p)=\zeta(p)-\langle\sigma\rangle_+\,p \tag{4.4}$$

for all p.

This theorem can be found in [GC95] for evolution maps and in [Ge98] for flows (which is the version we use here): see also [Ru99a]. The quantity p depends on $\underline{u}$. The quantity $\langle\sigma\rangle_+$ is also called "average entropy creation rate" and $p = p(\underline{u})$ is the dimensionless entropy creation rate averaged over a time τ and in the point $\underline{u}$: see [An82], [Ru96], [GR97], [Ru99a]. We recall that entropy in systems out of equilibrium is not defined (yet) so that this name needs not be taken too seriously and it might eventually reveal itself inappropriate.
The above result should not be confused (as it is conceptually and technically different) with other apparently similar statements, see [CG99]. It was discovered as an experimental relation in a numerical simulation, [ECM93], where the role of the SRB distributions and of time reversal were also suggested to be a possible reason for its validity: this led to its proof for Anosov maps in [GC95] and for flows in [Ge97].

A key feature of the theorem is that it contains *no free parameters*: its generality makes it a mechanical identity in the same sense, although of course of not comparable importance, as the heat theorem of Boltzmann, [Bo66], [Bo84], see also [Ga99a].
In the case of axiom C systems (4.4) still holds, because the evolution restricted to $\mathcal{A}$ is transitive and reversible by (4.2), *but σ has to be replaced by the contraction rate σ_0 of the surface area of the attracting set $\mathcal{A}$*. The quantities σ and σ_0 seem unrelated; however there are important cases in which the total phase space contraction σ and the contraction of the surface element of the attracting set σ_0 are proportional: $\sigma_0 = \vartheta_0\sigma$ with ϑ_0 a constant factor (or varying on a slower time scale than σ itself). Then $\langle\sigma_0\rangle_+ = \vartheta\langle\sigma\rangle_+$ with $\vartheta = \langle\vartheta_0\rangle_+$ and (4.4) becomes

$$\zeta(-p) = \zeta(p) - \vartheta\,\langle\sigma\rangle_+ \tag{4.5}$$

for all p. The GNS case is not among the (important) cases in which σ and σ_0 are proportional, see [DM96], [WL98],[BGG97] and [BG97], although heuristic arguments can be given, [Ga97a], suggesting that nevertheless a relation like (4.5) might hold.
In some cases in which proportionality between σ and σ_0 can be established, at least on heuristic grounds, the proportionality factor is just $1 - d(\mathcal{A})/d$ if d is the dimension of phase space and $d(\mathcal{A})$ is the dimension of the attractor, but unfortunately such cases are very rare, [BGG97], [BG97].

Finally it is worth writing explicitly the expression of the phase space contraction $\sigma(\underline{u})$ for the equation (3.7)

$$\begin{aligned}\sigma(\underline{u}) = {}&\Big(\sum_{|\underline{k}|<K(R)} \underline{k}^{\,2}\Big)\,\nu_G(\underline{u}) - \Big(\int_V \Delta\,\underline{\varphi}\cdot\Delta\,\underline{u}\;d\,\underline{x}\Big)\Big(\int_V [(\Delta\,\underline{u})^2 -\\ &- R\,\Delta\,\underline{u}\cdot(\Delta(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u})) - R(\Delta\underset{\sim}{u})\cdot(\Delta\,\underline{u})\cdot(\underset{\sim}{\partial}\,\underline{u}) - R\Delta\,\underline{u}\cdot(\Delta\underset{\sim}{\partial}\,\underline{u})\underset{\sim}{u}+\\ &+\nu(\underline{u})\Delta\,\underline{u}\cdot\Delta^2\,\underline{u}]d\,\underline{x}\Big)\;/\int_V(\Delta\,\underline{u})^2\,d\,\underline{x}\end{aligned} \tag{4.6}$$

In this expression (straightforwardly derived by imposing that $\mathcal{S}$ is exactly constant on motions verifying (3.7)) the first term seems to be the dominant one at large R so that

$$\sigma(\underline{u}) \simeq \Big(\sum_{|\underline{k}|<K(R)} \underline{k}^{\,2}\Big)\,\nu(\underline{u}) \stackrel{def}{=} \mathcal{K}(R)\,\nu_G(\underline{u}), \qquad \mathcal{K}(R) \propto R^{3\kappa_0+2} \tag{4.7}$$

which, if the side L_0 of the box is not $L = 1$ would be written with $\mathcal{K}(R)$ replaced by $\mathcal{K}_{L_0}(R) = \sum_{|\underline{k}|<K(R)} \underline{k}^{\,2}$ with $\underline{k} = 2\pi\,\underline{n}/L_0$.

§5. Fluctuation patterns and an extension of Onsager–Machlup fluctuations theory.

A physical interpretation of the fluctuation theorem, when it holds, can be found along with proposals for its test in experiments. We need first some consequences of the (technique of proof of the) fluctuation theorem.

Given an observable $H(\underline{u})$ we say that in its evolution observed over a time interval of size τ it *follows a pattern* $t \to h(t)$ if $F(S_t\,\underline{u}) = h(t)$ for $t \in [-\tau/2, \tau/2]$. We assume that F has well defined time reversal parity $\varepsilon = \pm 1$: $F(I\,\underline{u}) = \varepsilon\,F(\underline{u})$, for simplicity; and we say that the pattern $Ih(t) = \varepsilon\,h(-t)$ is the *time reversed pattern* of h.

Fluctuation patterns are the main object of analysis in the theory of Onsager–Machlup which deals with the probability of observing a fluctuation pattern h for an observable in the linear response regime (*i.e.* strictly speaking it deals with derivatives of various quantities with respect to the strength of the forcing terms evaluated at 0 forcing).

The following theorem can be regarded a result of the same type *without the restriction* that the system is in the linear response regime.

Theorem *(entropy creation as a fluctuations driver):*
Consider a time reversible evolution verifying the chaotic hypothesis and transitivity. Let H, K be two local observables (of given time reversal parity) and denote $\mu_{R,\tau,p}$ the SRB distribution conditioned to a (dimensionless) phase space average contraction p over a time span τ. Let h, k be two fluctuation patterns for H, K and let Ih, Ik be their time reversal patterns. Then if $\mu_{R,\tau,p}(\text{pattern of } H = h)$ denotes the probability that H follows the pattern h in the time span τ in which the average dimensionless phase space contraction is p it is

$$\frac{\mu_{R,\tau,p}(\text{pattern of } H = h)}{\mu_{R,\tau,p}(\text{pattern of K} = k)} = \frac{\mu_{R,\tau,-p}(\text{pattern of } H = Ih)}{\mu_{R,\tau,-p}(\text{pattern of K} = Ik)} \tag{5.1}$$

for large τ. If the system verifies axiom C the same result holds with IP (see (4.2) replacing I (without requiring any relation between the total phase space contraction rate and the rate of contraction of the surface of the attractor).

In other words the relative probability of fluctuation patterns of H and K observed in a time span τ during which the average entropy creation rate is $p\,\langle\sigma\rangle_+$ are the same as those of the time reversed patterns in a time span τ in which the average entropy creation rate is the opposite: $-p\,\langle\sigma\rangle_+$.

This allows us to give a physical interpretation to p: namely if we look at the evolution on time laps of size τ we see that the average entropy creation rate p will be usually $p = 1$ and the probability of observing $p \neq 1$ will be rare and the fraction of times we shall observe it is $e^{(\zeta(p)-\zeta(1))\tau}$: hence events in which $p \neq 1$ will be rare and random (*i.e. intermittent*) and they will take place at rate $\zeta(1) - \zeta(p)$.

The above theorem shows that when p is significantly different from 1 "*things go very wrong*". The frequency of findings of a time interval of size τ during which the time reversed patterns are relatively as probable as the normal patterns will be given by $e^{-\langle\sigma\rangle_+\,\tau}$ *no matter which observable H we look at*: an independence property that can be checked, in principle, in an experiment.

Hence a physical interpretation of p is that it is a quantitative measurement of the degree of reversibility that is observed. The larger $1 - p$ is the more "*unintuitive behavior*" will be observed. For $p = -1$ everything would be dramatically different from what expected.[1] The time intervals during which anomalous behavior is observed are rare so

[1] "If entropy creation rate could be changed in sign for a minute around Niagara falls then during that

that their manifestation is intermittent and we call this phenomenon "*entropy driven intermittency*": the function $\zeta(p)$ describes the phenomenon quantitatively.

§6. Entropy driven intermittency. Observability.

We now address the question: "is this intermittency observable"? is its rate function $\zeta(p)$ measurable?

Clearly σ_+ and $\zeta(p)$ will grow with the size of the system *i.e.* with the number of degrees of freedom, at least, which $\xrightarrow[R\to\infty]{}\infty$ so that there should be serious doubts about the observability of so rare fluctuations.

However if we look at a small subsystem in a little volume V_0 of linear size L_0 we can regard it again as a fluid enclosed in a box V_0 described by the same reversible GNS equations. We can imagine, therefore, that this small system also verifies a fluctuation relation in the sense that if, *c.f.r.* (4.7), (3.3)

$$\begin{aligned}&\sigma_{V_0}(\underline{u}) = \mathcal{K}_{L_0}(R)\,\nu_G(\underline{u})\\&\nu_G(\underline{u}) = \frac{\int_{V_0}\big(\underline{\varphi}\cdot\Delta\,\underline{u} - R\Delta\,\underline{u}\cdot(\underline{u}\cdot\underline{\partial}\,\underline{u})\big)\,d\,\underline{x}}{\int_{V_0}(\Delta\,\underline{u})^2\,d\,\underline{x}}\end{aligned}\tag{6.1}$$

then it should be that the fluctuations of σ averaged over a time span τ are controlled by rate functions $\zeta_V(p)$ and $\zeta_{V_0}(p)$ that we can expect to be, for R large

$$\begin{aligned}&\zeta_V(p) = \overline{\zeta}(p)\,\mathcal{K}_L(R), \qquad \text{and}\\&\zeta_{V_0}(p) = \overline{\zeta}(p)\,\mathcal{K}_{L_0}(R),\\&\langle\sigma_{V_0}\rangle_+ = \overline{\sigma}_+\,\mathcal{K}_{L_0}(R)\end{aligned}\tag{6.2}$$

We recall that, *c.f.r.* (4.7), $\mathcal{K}_{L_0}(R) = \sum_{|\underline{k}|<K(R_{L_0})}\underline{k}^2$ where R_{L_0} is the Reynolds number on scale L_0 which from the OK41 theory is $R_{L_0} = (L_0/L)^{4/3}R$. So that $\mathcal{K}_{L_0}(R) \propto (L_0/L)^3R^{15/4}$. Hence if we consider observables dependent on what happens inside V_0 and if L_0 is small so that $\mathcal{K}_{L_0}(R)$ is not too large and we observe them in time intervals of size τ then the time frequency during which we can observe a deviation "of size" $1-p$ from irreversibility will be small of the order of

$$e^{(\overline{\zeta}(p)-\overline{\zeta}(1))\,\tau\,\mathcal{K}_{L_0}(R)}\tag{6.3}$$

for τ large, where the local fluctuation rate $\overline{\zeta}(p)$ verifies (assuming transitivity or axiom C)

$$\overline{\zeta}(-p) = \overline{\zeta}(p) - \overline{\sigma}_+\,p\,\vartheta\tag{6.4}$$

with $\vartheta = 1$ in the transitive case and perhaps $\neq 1$ when the attracting set is smaller than phase space.

Therefore by observing the frequency of intermittency one can gain some access to the function $\overline{\zeta}(p)$.

Note that one *will necessarily observe a given fluctuation somewhere* in the fluid if L_0 is taken small enough: in fact the entropy driven intermittency takes place not only in time but also in space. Thus we shall observe inside a box of size L_0 "somewhere" in the total volume V of the system a fluctuation of size $1-p$ with high probability if

$$(L/L_0)^3e^{(\overline{\zeta}(p)-\overline{\zeta}(1))\,\tau\,\mathcal{K}_{L_0}(R)} \simeq 1\tag{6.5}$$

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minute their water would be more likely to go up rather than down". One "just" has to change the sign of the entropy creation rate, no extra effort needed!

and the special event $p = -1$ will occur with high probability if

$$(L/L_0)^3 e^{-\overline{\sigma}_+ \, \tau \, \mathcal{K}_{L_0}(R)} \simeq 1 \tag{6.6}$$

by (6.4). Once this event is realized the fluctuation patterns will have relative probabilities as described in §5.

An attempt at interpreting the experiment performed by Ciliberto and Laroche on convecting water at room temperature in terms of the above theory is in [Ga99d].

The idea and the possibility of local fluctuation theorems has been developed and tested first numerically, [GP99], and then theoretically, [Ga99c], by showing that it indeed works at least in some models (with homogeneous dissipation like the GNS and NS equations) which are simple enough to allow us to build a mathematically complete theory of the phase space contraction fluctuations.

Of course if the quantity $\langle\sigma_V\rangle_+$ and $p\,\langle\sigma_V\rangle_+$ could be measurable, like the equilibrium entropy, in terms of heat ceded to the various thermostats acting on the system divided by their temperature, then the whole theory could be even more easily subjected to experimental check as we could directly measure the rate function $\zeta(p)$ at least in small (but still macroscopically large) subvolumes. But the interpretation of the phase space contraction as a "physical entropy" (a concept that, however, we mentioned as *still requiring a definition* in non equilibrium physics) is quite controversial, *e.g.* see [Ho99] p. 236 and p. 240, as any statement about entropy is doomed to be.

I adhere to the point of view, [An82], [GR97], [Ru99a], that the *right definition* of entropy creation rate in systems out of equilibrium is just the phase space contraction rate: but the connection with measurable entities of the quantity so defined is (therefore) an open problem. Nevertheless we have seen that there are already quite a few checks to test the theory that are already possible (and necessary given the large number of assumptions that must be made to obtain them).

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